Solving Algebraic and Differential Riccati Operator Equations*

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ABSTRACT

By application of the Riesz-Dunford functional calculus a method for solving generalized algebraic Riccati operator equations in Hilbert space is given. Results are applied to the problem of finding explicit solutions for Cauchy problems and two-point boundary-value problems for generalized Riccati operator differential equations.

1. INTRODUCTION

Throughout this paper H denotes a complex separable Hilbert space and $L(H)$ denotes the algebra of all bounded operators on H. Cauchy problems for Riccati operator equations of the type

$$\frac{d}{dt}U(t) = A + BU(t) + U(t)C + U(t)DU(t), \qquad U(0) = U_0, \quad (1.1)$$

where A, B, C, D, and U_0 are operators in $L(H)$, arise in control theory [23], transport theory [26], and filtering problems [4]. The finite-dimensional case has been introduced in [18–19], and the infinite-dimensional case has been

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studied in [6, 20, 27]. In a recent paper [15], we studied the problem (1.1) and the two-point boundary-value problem

$$\frac{d}{dt}U(t) = A + BU(t) + U(t)C + U(t)DU(t),$$

$$EU(b) - U(0)F = G. \tag{1.2}$$

Conditions for the existence of solutions and explicit expressions for solutions are given in terms of the entries of the exponential function

$$S(t) = \exp\left(t\begin{bmatrix} C & D \\ A & B \end{bmatrix}\right);$$

but explicit expressions for the entries of $S(t)$ are unknown. In this paper we show that from the existence of a solution for the corresponding algebraic Riccati equation

$$A + BX + XC + XDX = 0 \tag{1.3}$$

the problems (1.1) and (1.2) may be transformed into others for which explicit expressions for solutions are available.

The problem of solving an operator equation of the type (1.3) appears in the study of certain control problems [31, 30, 6, 29] and the quadratic eigenvalue problem [9]. It has been studied by several authors in different contexts, [5, 7, 22, 13, 24, 29, 14]. In Section 2 of this paper we present a functional method for solving Equation (1.3) that permits one to reduce this equation to a system of two linear operator equations, and under certain conditions to only one linear operator equation. This reduction is available by means of the application of the Riesz-Dunford functional calculus and considering annihilating analytic functions of operators. It is clear that any finite-dimensional operator is annihilated by a polynomial. For the infinite-dimensional case this is not true, and operators which are annihilated by a polynomial are called algebraic operators. In [11], P. R. Halmos proved that an operator that is annihilated by an entire function must be algebraic. Under certain conditions contained in the minimal theorem [8, p. 571], the Riesz-Dunford functional calculus provides annihilating analytic functions of certain operators. If U denotes the open unit disc in the complex plane, and H^∞ denotes the Banach algebra of all bounded analytic functions on U under the supremum norm, then for the class C_0 of completely nonunitary contractions

W, there exists a nonzero function $f \in H^\infty$ such that $f(T) = 0$, where $f(T)$ means the image of T by f through the Nagy-Foiaş functional calculus [25].

We recall that if T lies in $L(H)$, the set of all complex numbers λ such that the range of $T - \lambda I$ is not dense is called the compression spectrum $\sigma_{\text{comp}}(T)$; a complex number λ is said to be an eigenvalue of T if $T - \lambda I$ is not injective, and we represent by $\sigma_p(T)$ the set of all eigenvalues of T. It is known (see [3, p. 240]) that

$$\sigma_p(T) = \{\lambda \in \mathbb{C};\, T - \lambda I \text{ is a left divisor of zero in } L(H)\},$$

$$\sigma_{\text{comp}}(T) = \{\lambda \in \mathbb{C};\, T - \lambda I \text{ is a right divisor of zero in } L(H)\}.$$

The paper is organized in the following way. Section 2 is concerned with the solution of the problem (1.3) by reducing Equation (1.3) to a system of the type $M + NX = 0$, $E + FX = 0$. Under certain additional conditions Equation (1.3) may be reduced to only one linear operator equation of the type $E + FX = 0$. Results of Section 2 are applied in Section 3 to the problem of finding explicit solutions for the problems (1.1) and (1.2). Because of the relation between algebraic and differential problems this paper may be regarded as a continuation of [12, 15, 16], and [17].

2. SOLVING THE ALGEBRAIC RICCATI OPERATOR EQUATION $A + BX + XC + XDX = 0$

We start this section with a characterization of solvability for Equation (1.3) that permits a functional treatment of that equation and that reduces the problem (1.3) to a system of two linear operator equations.

THEOREM 1. *Let f be an analytic function on an open set Ω containing the spectrum $\sigma(W)$, where*

$$W = \begin{bmatrix} B & -A \\ D & -C \end{bmatrix}, \qquad f(W) = \begin{bmatrix} M & E \\ N & F \end{bmatrix}. \tag{2.1}$$

(i) *Let X be an operator in $L(H)$. Then X is a solution of Equation* (1.3) *if and only if X satisfies the equation*

$$(B + XD)[I, X] = [I, X]W. \tag{2.2}$$

(ii) *If* $X \in L(H)$ *is a solution of* (1.3) *such that* $f(B+XD)=0$, *then* X *satisfies the operator system*

$$\begin{aligned} M + XN &= 0, \\ E + XF &= 0. \end{aligned} \tag{2.3}$$

(iii) *If* $X \in L(H)$ *is a solution of the system* (2.3) *and* $0 \notin \sigma_{\text{comp}}(N)$, *then* X *satisfies* (1.3) *and* $f(B+XD)=0$.

(iv) *If* $X \in L(H)$ *is a solution of the system* (2.3) *and* $0 \notin \sigma_{\text{comp}}(F)$, *then* X *satisfies* (1.3) *and* $f(B+XD)=0$.

Proof. (i): Computing both sides of the equality (2.2), we have

$$[I, X]W = [B + XD, -A - XC],$$

$$(B+XD)[I,X] = [B+XD, BX + XDX].$$

Hence the result is proved.

(ii): If X is a solution of Equation (1.3), then from (i), X satisfies (2.2). Postmultiplying (2.2) by the operator W, one gets

$$[I,X]W^2 = (B+XD)[I,X]W,$$

and by application of (2.2) again, one gets

$$[I,X]W^2 = (B+XD)^2[I,X].$$

Recurrently, for any positive integer n, we have

$$[I,X]W^n = (B+XD)^n[I,X]. \tag{2.4}$$

If $f(z) = \sum_{n \geqslant 0} a_n z^n$ is the power-series expansion of f, by application of the Riesz-Dunford functional calculus to the operators W and $B + XD$, and taking into account (2.4), it follows that

$$[I,X]f(W) = f(B+XD)[I,X], \tag{2.5}$$

$$[I,X]\begin{bmatrix} M & E \\ N & F \end{bmatrix} = f(B+XD)[I,X]. \tag{2.6}$$

From the hypothesis it follows that

$$[I, X]\begin{bmatrix} M & E \\ N & F \end{bmatrix} = 0.$$

Thus X satisfies (2.3).

(iii): Let us suppose X satisfies (2.3). By equating the entries arising in the first column of the equality $f(W)W = Wf(W)$, one gets

$$ED = BM - MB - AN, \tag{2.7}$$

$$FD = DM - NB - CN. \tag{2.8}$$

From (2.7) and from the equation $E + XF = 0$, it follows that

$$(BM - MB - AN) + XFD = 0. \tag{2.9}$$

From (2.8) and (2.9) we have

$$(BM - MB - AN) + X(DM - NB - CN) = 0. \tag{2.10}$$

By substituting $M = -XN$ in the expression (2.10), it follows that

$$-(A + BX + XC + XDX)N = 0.$$

From the hypothesis $0 \notin \sigma_{\text{comp}}(N)$ the result is proved.

(iv): If X is a solution of (2.3), then by equating the entries arising in the second column of the equality $f(W)W = Wf(W)$, it follows that

$$EC = -MA - BE + AF,$$

$$FC = -NA - DE + CF. \tag{2.11}$$

Postmulitplying the equation $E + TF = 0$ by the operator C and considering (2.11), we have

$$EC + XFC = 0,$$

$$(-MA - BE + AF) + X(-NA - DE + CF) = 0,$$

$$(A + BX + XC + XDX)F = 0.$$

From the hypothesis $0 \notin \sigma_{\text{comp}}(F)$, we have that X satisfies Equation (1.3). ∎

By means of the resolution of the operator system (2.3), Theorem 1 permits one to obtain solutions X of Equation (1.3) such that $B + XD$ is annihilated by an analytic function f which defines the coefficient operators of (2.3). Thinking of applications to the finite-dimensional case, in the following corollary we give more concrete information about the class of analytic functions f providing solutions of the original equation (1.3).

COROLLARY 1 [10]. *Let us suppose H is an n-dimensional space, and let us use the notation of Theorem* 1.

(i) *If f is analytic function on Ω such that* $\operatorname{rank} f(W) > n$, *then no solution of Equation* (1.3) *satisfies* $f(B + XD) = 0$.

(ii) *If X is a solution of* (1.3) *and $p(z)$ is an annihilating polynomial of the matrix $B + XD$, then X satisfies* (2.3), *where the coefficient operators are related to $p(z)$ by* (2.1).

(iii) *If f is an analytic function on Ω such that* $\operatorname{rank} f(W) = n$ *and F is an invertible matrix, then we have $M = EF^{-1}N$. Also, the system* (2.3) *has only one solution $X = -EF^{-1}$, and this solution is a solution of Equation* (1.3).

(iv) *If f is an analytic function on Ω such that* $\operatorname{rank} f(W) = n$ *and N is invertible, satisfying $E = MN^{-1}F$, then $X = -MN^{-1}$ is a solution of the system* (2.3) *that satisfies the original equation* (1.3).

Proof. (i): If there exists a solution X of Equation (1.3), let f be an annihilating polynomial of the matrix $B + XD$. Then from (2.5) one gets $[I, X]f(W) = 0$, and thus it is necessary that $\operatorname{rank} f(W) \leqslant n$.

(ii): This is a consequence of Theorem 1(ii).

(iii): From the invertibility of F, considering (2.1) and taking into account the following decomposition:

$$f(W) = \begin{bmatrix} M & E \\ N & F \end{bmatrix} = \begin{bmatrix} I & EF^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} M - EF^{-1}N & 0 \\ 0 & F \end{bmatrix} \begin{bmatrix} I & 0 \\ F^{-1}N & I \end{bmatrix}. \quad (2.12)$$

From the invertibility of the matrices arising in the first and third factors of the right hand side of (2.12), the condition $\operatorname{rank} f(W) = n$ implies that $M - EF^{-1}N = 0$. Now the result is a consequence of Theorem 1(iv).

(iv): This is a consequence of Theorem 1(iii). ∎

The next result is an analogous and alternative approach to the one developed in Theorem 1. It is based on a different characterization for the solvability of Equation (1.3).

THEOREM 2. *Let g be an analytic function on an open set Ω containing $\sigma(V)$ where*

$$V = \begin{bmatrix} -B & -A \\ D & C \end{bmatrix}, \qquad f(V) = \begin{bmatrix} P & Q \\ R & S \end{bmatrix}. \tag{2.13}$$

(i) *If $X \in L(H)$, then X is a solution of Equation* (1.3) *if and only if X satisfies the equation*

$$V\begin{bmatrix} X \\ I \end{bmatrix} = \begin{bmatrix} X \\ I \end{bmatrix}(C + DX). \tag{2.14}$$

(ii) *If $X \in L(H)$ is a solution of Equation* (1.3) *such that $g(C + DX) = 0$, then X satisfies*

$$PX + Q = 0,$$
$$RX + S = 0. \tag{2.15}$$

(iii) *If $X \in L(H)$ satisfies the system* (2.15) *and $0 \notin \sigma_p(P)$, then X is a solution of* (1.3) *and $g(C + DX) = 0$.*

(iv) *If $X \in L(H)$ is a solution of* (2.15) *and $0 \notin \sigma_p(R)$, then X satisfies* (1.1) *and $g(C + DX) = 0$.*

Proof. This is exactly dual to Theorem 1. ■

COROLLARY 2 [10]. *Let H be an n-dimensional space, and let us use the notation of Theorem* 2.

(i) *If g is an analytic function on Ω such that* $\operatorname{rank} g(V) > n$, *then no solution of Equation* (1.3) *satisfies $g(C + DX) = 0$.*

(ii) *If X is a solution of Equation* (1.3) *and $q(z)$ is an annihilating polynomial of the matrix $C + DX$, then X is a solution of the system* (2.15) *where the coefficient operators of* (2.15) *are given by* (2.13), *taking $g = q$.*

(iii) *If g is an analytic function on Ω such that* $\operatorname{rank} g(V) = n$, *and P is invertible with $S = RP^{-1}Q$, then $X = -P^{-1}Q$ is the only solution of* (2.15) *and satisfies* (1.3).

(iv) *If g is an analytic function on Ω such that* $\operatorname{rank} g(V) = n$, *and R is invertible with $Q = PR^{-1}S$, then the only solution of* (2.15) *is given by $X = -R^{-1}S$, and X is a solution of* (1.3).

Proof. This is a consequence of Theorem 2. ■

3. RICCATI-OPERATOR DIFFERENTIAL PROBLEMS

In Section 2 we have presented a method for obtaining explicit solutions of algebraic operator equations of the type (1.3). Now, let us consider the Cauchy problem

$$\frac{d}{dt}X(t) = A + BX(t) - X(t)C - X(t)DX(t), \qquad X(0) = P_0, \quad (3.1)$$

where A, B, C, D, and P_0 are operators in $L(H)$, and t belongs to an interval J such that $0 \in J$. Let us suppose that there exists a solution X_0 of the algebraic equation

$$A + BX - XC - XDX = 0. \quad (3.2)$$

Then, considering the change $U(t) = X(t) - X_0$, the problem (3.1) is equivalent to the problem

$$\frac{d}{dt}U(t) = B_0 U - UC_0 - UDU, \qquad U(0) = U_0, \quad (3.3)$$

where

$$B_0 = B - X_0 D, \qquad C_0 = C + DX_0, \qquad U_0 = P_0 - X_0. \quad (3.4)$$

Let us consider the extended linear system

$$\frac{d}{dt}\begin{bmatrix} V(t) \\ Z(t) \end{bmatrix} = \begin{bmatrix} C_0 & D \\ 0 & B_0 \end{bmatrix}\begin{bmatrix} V(t) \\ Z(t) \end{bmatrix}, \qquad \begin{bmatrix} V(0) \\ Z(0) \end{bmatrix} = \begin{bmatrix} I \\ U_0 \end{bmatrix}. \quad (3.5)$$

We define the operator function $S:]-\infty, +\infty[\times]-\infty, +\infty[\to L(H \oplus H)$ by the expression

$$S(t,s) = \begin{bmatrix} \exp[(t-s)C_0] & \int_0^t \exp[(t-v)C_0]\, D \exp[(v-s)B_0]\, dv \\ 0 & \exp[(t-s)B_0] \end{bmatrix}. \quad (3.6)$$

Computing, it is a straightforward matter to show that

$$\frac{\partial}{\partial t}S(t,s)=\begin{bmatrix} C_0 & D \\ 0 & B_0 \end{bmatrix} S(t,s), \tag{3.7}$$

and thus $S(t,s)$ is a fundamental operator for (3.5), and the only solution of (3.5) is given by (see [21] for details)

$$\begin{bmatrix} V(t) \\ Z(t) \end{bmatrix} = S(t,0)\begin{bmatrix} I \\ U_0 \end{bmatrix}. \tag{3.8}$$

Note that $V(0)=I$, and thus in a neighborhood J_1 containing the origin, $V(t)$ is invertible in $L(H)$. Now, let us define the operator-valued function $U(t)=Z(t)[V(t)]^{-1}$ for all t in J_1. Note that from (3.5) it follows that $(d/dt)V(t)=C_0V(t)+DZ(t)$ and $(d/dt)Z(t)=B_0Z(t)$. By computing it follows that

$$\begin{aligned}\frac{d}{dt}U(t)&=\left(\frac{d}{dt}Z(t)\right)[V(t)]^{-1}-Z(t)[V(t)]^{-1}\left(\frac{d}{dt}V(t)\right)[V(t)]^{-1}\\ &=B_0U(t)-U(t)C_0-U(t)DU(t), \qquad t\in J_1.\end{aligned}$$

Hence $X(t)=U(t)+X_0$ is a solution of the problem (3.1) defined on the interval J_1. From the expression (3.7) we have

$$\begin{aligned}V(t)&=\exp(tC_0)+\int_0^t\exp((t-v)\,C_0D\exp(vB_0)\,dv\,U_0\\ &=\exp(tC_0)\left(I+\int_0^t\exp(-vC_0)\,D\exp(vB_0)\,dv\,U_0\right),\\ Z(t)&=\exp(tB_0)\,U_0,\\ X(t)&=X_0+\exp(tB_0)\,U_0\left(I+\int_0^t\exp(-vC_0)\,D\exp(vB_0)\,dv\,U_0\right)^{-1}\\ &\qquad\times\exp(-tC_0)\end{aligned} \tag{3.9}$$

for all $t\in J_1$. The following result has been proved.

THEOREM 3. *Let us suppose $X_0 \in L(H)$ is a solution of the algebraic operator equation* (3.2), *and let us consider the Cauchy problem* (3.1). *Then there exists an interval J_1 containing the origin such that the only solution of* (3.1) *is given by* (3.9), *where B_0, C_0, and U_0 are defined by* (3.4).

Now, let us consider the boundary-value problem

$$\frac{d}{dt}X(t) = A + BX(t) - X(t)C - X(t)DX(t),$$

$$EX(b) - X(0)F = G, \quad 0 \leqslant t \leqslant b \quad (3.10)$$

Considering the change $X(t) = U(t) + X_0$, where X_0 is a solution of Equation (3.2), the problem (3.10) is equivalent to the problem

$$\frac{d}{dt}U(t) = B_0U(t) - U(t)C_0 - U(t)DU(t),$$

$$EU(b) - U(0)F = G - EX_0 + X_0F. \quad (3.11)$$

Let us define $S(t) = S(t,0)$, where S is defined by (3.6) and t lies in the real line, from Theorem 1 and Theorem 2 of [15], if we denote $U_1 = U(0)$, in order to satisfy the boundary-value condition of (3.11), a solution of the problem

$$\frac{d}{dt}U(t) = B_0U(t) - U(t)C_0 - U(t)DU(t), \qquad U(0) = U_1, \quad (3.12)$$

satisfying

$$\exp(tC_0)\left(I + \int_0^t \exp[(t-v)C_0]\, D\exp(vB_0)\, dv\, U_1\right)$$

is invertible for $t \in [0, b]$

must verify the algebraic equation

$$M + NX - XP - XQX = 0, \quad (3.13)$$

where the operators M, N, P, and Q are given by the expressions

$$M = -(G - EX_0 + X_0 F)\exp(bC_0),$$

$$Q = F\int_0^b \exp[(b-v)C_0]\, D\exp(vB_0)\, dv,$$

$$N = E\exp(bB_0) - (G - EX_0 + X_0 F)\left(\int_0^b \exp[(b-v)C_0]\, D\exp(vB_0)\, dv\right),$$

$$P = F\exp(bC_0) \tag{3.14}$$

(see [15] for details).

Let us suppose the algebraic equation (3.13)–(3.14) has a solution X_1 such that

$$I + \int_0^t \exp[(t-v)C_0]\, D\exp(vB_0)\, dv\, X_1 \text{ is invertible for } t \in [0,b]. \tag{3.15}$$

Then from [15, Theorem 2], the following expression defines a solution of the boundary-value problem (3.10):

$$X(t) = X_0 + \exp(tB_0)\, X_1\left(I + \int_0^t \exp(-vC_0)\, D\exp(vB_0)\, dv\, X_1\right)^{-1}$$

$$\times\exp(-tC_0), \tag{3.16}$$

where $B_0 = B - X_0 D$, $C_0 = C + DX_0$, X_0 is a solution of (3.2), and X_1 is a solution of (3.13)–(3.14) which satisfies (3.15). So the following result is proved.

THEOREM 4. *Let us suppose that X_0 is a solution of* (3.2), *and that X_1 is a solution of* (3.13)–(3.14) *and* (3.15), *where B_0, C_0 are given by* (3.4). *Then $X(t)$ defined by* (3.16) *is a solution of the boundary-value problem* (3.10).

REMARK 1. It is interesting to note that the problem (3.1) is always locally solvable (see [15]) and Equation (3.2) may be unsolvable. In that case the method provided by Theorem 3 is not applicable, but the condition (2.8) of [15, Theorem 2], and the expression for solutions of (3.1) provided in [15], are difficult to apply because the entries of

$$S(t,s) = \exp\left(\begin{bmatrix} C & D \\ A & B \end{bmatrix}(t-s)\right)$$

are not expressible in terms of the data of the problem. Analogously, [15, Theorem 2] yields a solution of (3.10) that only requires a solution of an equation of the type (3.13), but this solution is difficult to compute for the same reason.

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